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LANG MAPS AND HARRIS'S CONJECTURE

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ABSTRACT

We show that any variety in characteristic 0 possesses a universal dominant rational map, which we call **the Lang map**, to a variety of general type. We discuss a conjecture of J. Harris regarding the relation between rational points and Lang maps.

0. Introduction

We work over fields of characteristic 0.

Let X be a variety of general type defined over a number field K. A well known conjecture of S. Lang [L] states that the set of rational points $X(K)$ is not Zariski-dense in X. As noted in [ℵ], this implies that if X is a variety which only *dominates* a variety of general type then $X(K)$ is still not dense in X.

J. Harris proposed a way to quantify this situation [H1]: define the **Lang dimension** of a variety to be the maximal dimension of a variety of general type which it dominates. Harris conjectured in particular that if the Lang dimension is 0, then for *some* number field $L \supset K$ we have that the set of L rational points $X(L)$ is dense in X. The full statement of Harris's conjecture will be given below (Conjecture 2.3).

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The purpose of this note is to provide a geometric context for Harris's conjecture, by showing the existence of a universal dominant map to a variety of general type, which we call **the Lang map.**

1. The Lang map

THEOREM 1.1: *Let X be an irreducible variety over a field k,* $\mathrm{char}(k) = 0$. *There exists a variety of general type $L(X)$ and a dominant rational map $L : X \dashrightarrow L(X)$, defined over k, satisfying the following universal property:*

Given a field $K \supset k$ and a dominant rational map $f : X_K \dashrightarrow Z$ defined over K, where Z is of general type, then there exists a unique dominant rational map $L(f) : L(X) \dashrightarrow Z$ such that $L(f) \circ L = f$.

Definition 1.2: The universal dominant rational map $L : X \dashrightarrow L(X)$ is called **the Lang map***. The dimension $\dim L(X)$ is called **the Lang dimension** of X.

LEMMA 1.3: *Assume $f_i : X_K \dashrightarrow Z_i$, $i = 1, 2$ are dominant rational maps with irreducible general fiber, where Z_i are varieties of general type. Then there exists a variety of general type Z over K and dominant rational maps $f : X \dashrightarrow Z$ and $g_i : Z \dashrightarrow Z_i$ such that $g_i f = f_i$.*

Proof: Let $Z = \mathrm{Im}(f_1 \times f_2 : X \dashrightarrow Z_1 \times Z_2)$, and let $f : X \dashrightarrow Z$ be the induced map. The map $g_i : Z \dashrightarrow Z_i$ is dominant and has irreducible general fiber. We claim that Z is a variety of general type. By Viehweg's additivity theorem ([V1], Satz III), it suffices to show that the generic fiber of g_1 is of general type. This follows since the fibers of g_1 sweep Z_2. (Specifically, let d be the dimension of the generic fiber of g_2. Choose a general codimension-d plane section $H \subset Z_1$; then $g_1^{-1}H \to Z_2$ is generically finite and dominant, therefore $g_1^{-1}H$ is of general type, therefore the generic fiber of $g_1^{-1}H \to Z_1$ is of general type.) ∎

LEMMA 1.4: *Given a field extension $K \supset k$, let l_K be the maximal dimension of a variety of general type Y/K such that there exists a dominant rational map $L_K : X_K \dashrightarrow Y$ with irreducible general fiber. Let $l = \max_{K \supset k} l_K$, and let $K \supset k$*

* I believe this name is appropriate since (1) L is closely related to Lang's conjecture, and (2) the construction resembles in many ways the constructions of Albanese, trace etc. in Lang's book [L:AV].

be an extension such that $l_K = l$. Then any such map L_K is the Lang map of X_K.

Proof: Given an extension $E \supset K$ let $f_2 : X_E \dashrightarrow Z_2$ be a dominant rational map, where Z_2 is of general type. By the lemma above with $L_K = f_1$ there exists a variety Z of general type and a dominant rational map with irreducible general fiber $f: X_E \dashrightarrow Z$ dominating both Y_E and Z_2. By maximality $\dim Y = \dim Z_2$ and since the general fibers of $Z \dashrightarrow Y_E$ are irreducible, $Z \dashrightarrow Y_E$ is birational. The map $g_2 \circ g_1^{-1} : Y_E \dashrightarrow Z_2$ gives the required dominant rational map. ∎

Proof of the theorem: Using Stein factorization we may restrict attention to maps with irreducible general fibers. As above, let $l = \max_{K \supset k} l_K$, and let $K \supset k$ be an extension such that $l_K = l$. We need to show that L_K can be descended to k.

First, we may assume that K is finitely generated over k, since both Y and L_K require only finitely many coefficients in their defining equations.

Next, we descend L_K to an algebraic extension of k. Choose a model B for K, and a model $\mathcal{Y} \to B$ for Y. We have a dominant rational map $X_B \dashrightarrow \mathcal{Y}$ over B. There exists a point $p \in B$ with $[k(p) : k]$ finite, such that $\mathcal{Y}_p$ is a variety of general type of dimension l and such that the rational map $X_p \dashrightarrow \mathcal{Y}_p$ exists. The lemma above shows that $(\mathcal{Y}_p)_K$ is birational to Y. Alternatively, this step follows since by theorems of Maehara (see [Mor]) and Kobayashi–Ochiai (see [MD-LM]) the set of rational maps to varieties of general type $X \dashrightarrow Z$ is discrete, therefore each $f : X_K \dashrightarrow Z$ is birationally equivalent to a map defined over a finite extension of k.

We may therefore replace K by an algebraic Galois extension of k, which we still call K. Let $\mathrm{Gal}(K/k) = \{\sigma_1, \dots, \sigma_m\}$. For any $1 \leq i \leq m$ we have a rational map $(f_1 \times \sigma_i \circ f_1) : X \dashrightarrow Y \times Y^{\sigma_i}$. Applying Lemma 1.3 we obtain a birational map $Y \dashrightarrow Y^{\sigma}$. There are open sets $U_i \subset Y^{\sigma_i}$ over which these maps are regular isomorphisms, giving rise to descent data for U_1 to k. ∎

Is there a way to describe the fibers of the Lang map $X \dashrightarrow L(X)$? A first approximation is provided by the following:

PROPOSITION 1.5: *The generic fiber of the Lang map has Lang dimension* 0.

Proof: Let $\eta \in L(X)$ be the generic point and let $X_\eta \dashrightarrow L(X_\eta)$ be the Lang map of the generic fiber. Let $M \to L(X)$ be a model of $L(X_\eta)$. By definition, the generic fiber $M_\eta = L(X_\eta)$ of M is of general type, therefore by Viehweg's additivity theorem M is of general type, and by definition M is birational to $L(X)$. ∎

QUESTION 1.6: *Is there an open set in X where the Lang map is defined and the fibers have Lang dimension 0?*

We will see that the answer is yes, if one assumes the following inspiring conjecture of higher-dimensional classification theory:

CONJECTURE 1.7 (see Conjecture 1.24 of [Ko:FlAb]):

(1) *Let X be a variety in characteristic 0. Then either X is uniruled, or* $\mathrm{Kod}(X) \geq 0$.
(2) *If* $\mathrm{Kod}(X) \geq 0$, *then there is an open set in X where the fibers of the Iitaka fibration have Kodaira dimension 0.*

This conjecture allows us to "construct" the Lang map "from above":

PROPOSITION 1.8: *Assume that Conjecture 1.7 holds true. Then there is a finite sequence of dominant rational maps*

$$X \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n = L(X)$$

where each map $X_i \dashrightarrow X_{i+1}$ is either an MRC fibration (see [Ko-Mi-Mo], 2.7) *or an Iitaka fibration. In particular, the answer to Question* 1.6 *is "yes".*

Proof: Apply induction to $d = \dim X$. If $\mathrm{Kod}(X) < 0$, then X is uniruled, therefore the MRC fibration $f : X \dashrightarrow X_1$ is nontrivial. Since the fibers of f are rationally connected, we have that $L : X \dashrightarrow L(X)$ factors through f. If $\mathrm{Kod}(X) \geq 0$ let $f : X \dashrightarrow X_1$ be the Iitaka fibration. Since the generic fiber of f has Kodaira dimension 0, we have that L factors through f in this case as well. The map f is trivial only when X is of general type. The induction hypothesis on X_1 gives the result. ∎

We remark that 1.7 is known when the fibers have dimension ≤ 2. In particular, 1.8 is known unconditionally when $\dim X \leq 3$.

2. Harris's conjecture

As mentioned above, we define **the Lang dimension** of a variety X to be $\dim L(X)$, and Lang's conjecture implies that if K is a number field, and if X/K has positive Lang dimension, then $X(K)$ is not Zariski-dense in X. In [H1], J. Harris proposed a complementary statement:

CONJECTURE 2.1 (Harris's conjecture, weak form): *Let X be a variety of Lang dimension 0 defined over a number field K. Then for some finite extension $E \supset K$ the set of E-rational points $X(E)$ is Zariski-dense in X.*

It is illuminating to consider the motivating case of an elliptic surface of positive rank.

Let $\pi_0 : X_0 \to \mathbb{P}^1$ be a pencil of cubics through 9 rational points in $\mathbb{P}^2$. By choosing the base points in general position we can guarantee that the pencil has 12 irreducible singular fibers which are nodal rational curves. The Mordell–Weil group of π_0 has rank 8. The relative dualizing sheaf is $\omega_{\pi_0} = \mathcal{O}_{X_0}(F_0)$ where F_0 is a fiber. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a map of degree at least 3. Let $\pi : X \to \mathbb{P}^1$ be the pull-back of X_0 along f. Then $\omega_\pi = \mathcal{O}_X(3F)$, therefore $\omega_X = \mathcal{O}_X(F)$ and X has Kodaira dimension 1. The Iitaka fibration is simply π. The elliptic surface X still has a Mordell–Weil group of rank 8 of sections. By applying these sections to rational points on $\mathbb{P}^1$ we see that the set of rational points $X(\mathbb{Q})$ is dense in X.

It is not hard to modify this example to obtain a varying family of elliptic surfaces which has a dense collection of sections. Let B be a curve and let $g : B \times \mathbb{P}^1 \to \mathbb{P}^1$ be a family of rational functions on $\mathbb{P}^1$ which varies in moduli (such families exist as soon as the degree is at least 3). Let Y be the pullback of X to $B \times \mathbb{P}^1$. Then $p : Y \to B$ is a family of elliptic surfaces, of variation $\mathrm{Var}(p) = 1$, and relative Kodaira dimension 1. By composing sections of E with g and arbitrary rational maps $B \to \mathbb{P}^1$, we see that p has a dense collection of sections.

Harris's weak conjecture for elliptic surfaces is attributed to Manin. Recently, it has been related to the conjecture of Birch and Swinnerton-Dyer. For example, let $\pi : X \to \mathbb{P}^1$ be an elliptic surface defined over $\mathbb{Q}$. In [Man], E. Manduchi shows that under certain assumptions on the behavior of the j function, the set of points in $\mathbb{P}^1(\mathbb{Q})$ where the fiber has root number -1 is dense in the classical topology. According to the conjecture of Birch and Swinnerton-Dyer, the root number gives the parity of the Mordell–Weil rank. More recently G. Grant and E. Manduchi have obtained a much more general result (see [G-M]).

What can be said in case $0 < \dim L(X) < \dim X$? In [H2], Harris proposed the following definition:

Definition 2.2: The **diophantine dimension,** $\mathrm{Ddim}(X)$, is defined as follows:

$$\mathrm{Ddim}(X) := \min_{\emptyset \neq U \subset X \text{open}} \max_{[E:K]<\infty} \dim(\overline{U(E)}).$$

Harris proceeded to propose the following:

CONJECTURE 2.3 (Harris's conjecture): *For any variety X over a number field,*

$$\mathrm{Ddim}(X) + \dim L(X) = \dim X.$$

I do not know whether or not Harris himself believes this conjecture. This does not really matter. What is appealing in this conjecture, apart from its "tightness", is that any evidence, either for or against it, is likely to be of much interest.

Note that Proposition 1.8 directly implies the following:

PROPOSITION 2.4: *Assuming* 1.7, *Lang's conjecture together with the weak form of Harris's conjecture* 2.1 *implies Harris's conjecture* 2.3.

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Added in Proof. A simple counterexample was recently found to the conjecture of Harris, as well as a well known conjecture of Mazur, by J. L. Colloit-Thélène, A. N. Skorobogatov and Sir Peter Swinnerton-Dyer. Their paper is due to appear in Acta Arithmetica. The point is that the diophantine dimension is invariant under étale covers, whereas the Lang dimension is not.

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